



Teacher-to-Teacher

Video Series
for Secondary Educators

TITLE: Limits

PRIMARY SUBJECT AREAS: Precalculus/Calculus

GRADE LEVELS: 11-12

OVERVIEW: In this lesson, the concept of limits will be developed graphically, numerically, and algebraically. An exploration activity (lab), problem set, BINGO game, and puzzle will be included.

APPROXIMATE DURATION: One 50 minute class period for the lab and problem set other activities may be used in a later class period as review.

LOUISIANA CONTENT STANDARDS:

<http://www/DOE/assessment/standards/MATH.pdf>

Algebra

A-3-H Using tables and graphs as tools to interpret algebraic expressions, equations, and inequalities

Patterns, Relations, and Functions

P-2-H Translating between tabular, symbolic, or graphic representations of functions

GLEs Addressed

Grade 10

27. Translate among tabular, graphical, and symbolic representations of patterns in real-life situations, with and without technology (P-2-H) (P-3-H) (A-3-H)

Grades 11/12

6. Analyze functions based on zeros, asymptotes, and local and global characteristics of the function (A-3-H)

EDUCATIONAL TECHNOLOGY GUIDELINES:

<http://www/DOE/LCET/curric/k12stand.pdf>

Technology Problem-Solving and Decision-Making Tools

- Students use appropriate technology resources for solving problems and making informed decisions.

Technology Productivity Tools

- Students use technology tools to enhance learning, increase productivity, and promote creativity.

INTERDISCIPLINARY CONNECTIONS:

N/A

OBJECTIVES:

1. Given the graph or equation of a function, the student will tell whether or not the function has a limit as x approaches a given value.
2. The student will determine the limit using a table of values, a graph, and algebraic simplification.

LESSON MATERIALS AND RESOURCES:

Investigation worksheet, Calculus or Precalculus textbook

TECHNOLOGY TOOLS AND MATERIALS:

Overhead calculator

Calculator for each student

BACKGROUND INFORMATION:

Students should be able to use the table and graph features of the calculator and they should be able to simplify rational expressions, using factoring, conjugates, and other algebraic techniques.

LESSON PROCEDURES:

1. Use the lab worksheet investigation. This may be done as a whole class activity with the teacher as the facilitator using the overhead calculator and guiding the class discussion. Or it may be done in small groups, allowing each group to proceed at its own pace. In both cases students should use their own calculators to work through the lab. It is important for students to investigate each limit numerically and graphically. Then the algebraic method can be used as well. Working through problems one and two with the entire class can help students get a good foundation in limits.
2. Have students write individual conclusions based on their findings in the lab. If students can write conclusions about the work it will verify their understanding of the topic.
3. Work sample problems with students. The problem set may be done with the whole group, in small groups, or individually.
4. Play the BINGO game as a review of limits.
5. Assign the puzzle as a review at the end of the study of limits.

ASSESSMENT PROCEDURES:

1. If the lab is done as a whole class activity, students are assessed as the lab progresses. If it is done in small groups, then the teacher should move around the class to make sure students are making appropriate conclusions and doing the work correctly.
2. Students should write individual conclusions at the end of the investigation. These can be discussed in groups and/or with the whole

- class. The written conclusions can be turned in for teacher evaluation as well.
3. A quiz on finding limits using tables, graphs, and algebra can be used as a formal assessment. Have students do one problem, using all three methods.
 4. Have students write and solve their own limit problems.
 5. The BINGO game and puzzle allow the students to assess their own knowledge of limits.

ACCOMMODATIONS/MODIFICATIONS:

N/A

REPRODUCIBLE MATERIALS:

Limit Investigation worksheet and Limit Problem Worksheet
Limits Bingo and Limits Puzzle

EXPLORATION AND EXTENSION:

N/A

LESSON DEVELOPMENT RESOURCES:

N/A

REFLECTIONS:

Students reach a clear understanding of limits by working through this investigation. Different students reach the understanding of limits through different methods. So doing a lab that uses numerical, graphical, and also algebraic methods is helpful. Most students have trouble with the algebra involved in finding a limit, but they usually understand the use of a table or graph to find the limit. This investigation helps students transition from graphs and tables to algebra.

Students generally have more success in learning and understanding a difficult concept when they are involved in the lesson. By asking students to use calculators and progress through the lab, they are engaged in the learning process and are forced to take responsibility for their own learning. I think they get a thorough understanding of the concept of limits as well as the procedure for finding them.

The BINGO game and puzzle let both the student and teacher assess the student's progress in a non-threatening situation. My students enjoy making a game of math!

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LAB: LIMITS OF FUNCTIONS

GOALS:

- ◆ To develop an intuitive understanding of the nature of limits
- ◆ To lay the foundation for the use of limits in calculus
- ◆ To evaluate limits graphically, numerically, and algebraically

PROCEDURE:

In this lab, we will study the behavior of a function f near a specified point. While this may be a straight forward process, it can also be very subtle. In some instances in calculus the process for finding a limit must be applied carefully. By gaining an intuitive feel for the idea of limits, you will be laying a solid foundation for success in calculus.

As you go through this lab, evaluate the limits by the method given: **numerically** (using table on calculator), **graphically**, or **algebraically**. You might also want to try to use another method as a check and to see how all 3 methods can be used.

1. Consider the function f defined by $f(x) = \frac{x^4 - 1}{x - 1}$.
 - a) By successive evaluation of f at $x = 1.8, 1.9, 1.99, 1.999,$ and $1.9999,$ what do you think happens to the values of f as x increases towards 2 ?
 - b) Do a similar evaluation of f for values of x slightly greater than 2 . Comment on your results.
As a shorthand, we will describe these results by writing $\lim_{x \rightarrow 2} f(x) = 15$.
 - c) Evaluate $f(2)$. Comment on this answer.
2. Use the function f as above; consider what happens as x approaches 1 .
 - a) Study this situation numerically as you did in parts a and b above. What are your conclusions? In particular, what is $\lim_{x \rightarrow 1} f(x)$?
 - b) What is $f(1)$? What is the difference in the 2 situations?
 - c) Factor $f(x)$ and simplify. Now substitute 1 for x . Is this consistent with your findings in part a?

3. Using your calculator, determine the values of these limits:

$$\frac{x^4 - 1}{x - 1} \quad \text{for } x < 1$$

$$\text{a) } \lim_{x \rightarrow 1} g(x) \quad \text{where } g(x) = \begin{cases} 17 & \text{for } x = 1 \\ 14 - \frac{10}{x} & \text{for } x > 1 \end{cases} \quad \text{b) } \lim_{x \rightarrow -3} \frac{x^3 + 27}{x + 3}$$

$$\text{c) } \lim_{x \rightarrow 0} \frac{\sin(10x)}{x} \quad \text{d) } \lim_{x \rightarrow 0} (1 + x)^{1/x}$$

4. Limits can sometimes fail to exist. Investigate the following limits and explain why you think each does not exist. Use your calculator to graph functions as needed. Why are these different from the functions previously considered? Hint: Consider limits from left side or right side only.

$$\text{a) } \lim_{x \rightarrow 2} \frac{x}{x - 2} \quad \text{b) } \lim_{x \rightarrow 2} \frac{1}{(x - 2)^2} \quad \text{c) } \lim_{x \rightarrow 0} \frac{|x|}{x} \quad \text{d) } \lim_{x \rightarrow \frac{\pi}{2}} \tan x \quad \text{e) } \lim_{x \rightarrow 0} \sin \frac{1}{x}$$

Limit Problems

Find each limit if it exists:

$$1. \lim_{x \rightarrow 3} \frac{5x + 11}{x + 1}$$

$$2. \lim_{x \rightarrow 2} \frac{3x^2 - x - 10}{x^2 - x - 2}$$

$$3. \lim_{x \rightarrow 5} \frac{\frac{1}{x} - \frac{1}{5}}{x - 5}$$

$$4. \lim_{x \rightarrow 3^+} \frac{1}{x - 3}$$

$$5. \lim_{x \rightarrow 2} \frac{\sqrt{x} - \sqrt{2}}{x - 2}$$

$$6. \lim_{x \rightarrow 1} \frac{x^3 + x - 2}{x - 1}$$

$$7. \lim_{x \rightarrow 2} \frac{x^3 - 8}{x - 2}$$

$$8. \lim_{x \rightarrow -2} \frac{x^4 - 16}{x + 2}$$

$$9. \lim_{x \rightarrow \infty} \frac{6x^2 + 2x - 17}{4x^2 + 9}$$

$$10. \lim_{x \rightarrow \infty} \frac{5x + 11}{x^2 + 1}$$

Sketch each piecewise function f and, for the indicated value of a , find each limit if it exists:

$$a) \lim_{x \rightarrow a^-} f(x)$$

$$b) \lim_{x \rightarrow a^+} f(x)$$

$$c) \lim_{x \rightarrow a} f(x)$$

$$11. f(x) = \begin{cases} 3x & \text{if } x \leq 2 \\ x^2 & \text{if } x > 2 \end{cases} \quad a = 2$$

$$12. f(x) = \begin{cases} \frac{x^2 + x}{x} & \text{if } x \neq 0 \\ 2 & \text{if } x = 0 \end{cases} \quad a = 0$$

Using a graphing calculator, verify each limit by:

a) creating a table of function values for x close to a

b) graphing and zooming in on the graph of f at a

$$13. \lim_{x \rightarrow 0} \frac{x}{\tan x} = 1$$

$$14. \lim_{x \rightarrow 3} \frac{x^3 + 2x^2 - 9x - 18}{x - 3} = 30$$

$$15. \lim_{x \rightarrow 0} \frac{x}{\sin x} = 1$$

Limits BINGO

$-\frac{1}{9}$	12	3	5	1
7	$\frac{9}{2}$	0	2	DNE (∞)
DNE not defined from left	$-\frac{3}{2}$	DNE ($-\infty$)	-2	4
DNE	6	-4	$-\frac{1}{4}$	10
$\frac{11}{2}$	-3	-1	-5	DNE left lim \neq right lim

Find each limit. Then locate your answer on the BINGO board and circle it (also write problem number in the square). Work problems in any order until you have circled 5 answers in a row – horizontally, vertically, or diagonally. Then shout **BINGO!**

1. $\lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3}$

2. $\lim_{x \rightarrow 3} \frac{x^2 - x - 6}{x - 3}$

3. $\lim_{x \rightarrow 4} \frac{x - 4}{\sqrt{x} - 2}$

4. $\lim_{x \rightarrow 3} 7$

5. $\lim_{x \rightarrow \frac{1}{2}} \frac{6x^2 + 5x - 4}{2x - 1}$

6. $\lim_{x \rightarrow -1} \left(\sqrt[3]{x} - \frac{2}{\sqrt[3]{x}} \right)^5$

7. $\lim_{x \rightarrow 2} \frac{x - 2}{\frac{1}{x} - \frac{1}{2}}$

8. $\lim_{x \rightarrow 2} \frac{x^3 - 8}{x - 2}$

9. $\lim_{x \rightarrow \frac{1}{2}} \frac{5x + 2}{2x}$

10. $\lim_{h \rightarrow 0} \frac{2 - \sqrt{4 + h}}{h}$

11. $\lim_{h \rightarrow 0} \frac{(5 + h)^2 - 25}{h}$

12. $\lim_{x \rightarrow 4^-} \frac{|x - 4|}{x - 4}$

13. $\lim_{x \rightarrow 4} \frac{|x - 4|}{x - 4}$

14. $\lim_{x \rightarrow 4^+} \sqrt{x - 4} - 5$

15. $\lim_{x \rightarrow 4^-} \sqrt{x - 4}$

16.

$$\lim_{x \rightarrow \infty} \frac{15x^2 - 2x + 3}{5x^2 - 7}$$

17. $\lim_{x \rightarrow \infty} \frac{\sqrt{4x^2 + 5}}{x - 3}$

18. $\lim_{x \rightarrow -\infty} \frac{\sqrt{4x^2 + 5}}{x - 3}$

19. $\lim_{x \rightarrow 4^+} \frac{1}{x - 4}$

20. $\lim_{x \rightarrow 4^-} \frac{1}{x - 4}$

21. $\lim_{h \rightarrow 0} \frac{(3 + h)^{-1} - 3^{-1}}{h}$

22. $\lim_{x \rightarrow \infty} \frac{4 - 3x^3}{2x^3 + 3x - 1}$

23. $\lim_{x \rightarrow \infty} \frac{10 - 3x}{(2x + 1)^3}$

24. $\lim_{x \rightarrow \infty} \tan x$

What did one math book say to the other?

A	B	C	D	E
F	G	H	I	J
K	L	M	N	O

Directions: Evaluate each limit and find your answer in one of the answer columns. Notice the letters next to the answer. Write these letters in the box that has the same letter as the exercise. Put the letters together to form words and you will find out the answer to the question!

A $\lim_{x \rightarrow 4^-} x - \sqrt{16 - x^2}$

D $\lim_{x \rightarrow 5} \frac{x^3 - 125}{x - 5}$

N $\lim_{x \rightarrow 9} \frac{x - 9}{\sqrt{x} - 3}$

G $\lim_{x \rightarrow 3} \frac{x^3 - x^2 - 5x - 3}{x - 3}$

J $\lim_{x \rightarrow 4} \frac{\frac{1}{x} - \frac{1}{4}}{x - 4}$

F $\lim_{x \rightarrow \infty} \frac{5x^2 - 3x + 1}{2 - 3x^2}$

M $\lim_{x \rightarrow 2} \frac{1}{(x - 2)^2}$

B $\lim_{x \rightarrow \frac{1}{2}^-} \frac{1}{2x - 1}$

L $\lim_{x \rightarrow \infty} \frac{6x}{\sqrt{9x^2 + 1}}$

E $\lim_{x \rightarrow 2^+} \frac{|x - 2|}{x - 2}$

H $\lim_{x \rightarrow 6} \sqrt{36 - x^2}$

O $\lim_{x \rightarrow \infty} \frac{5x}{2 - 3x^2}$

K $\lim_{h \rightarrow 0} \frac{8(x+h) - 8x}{h}$

C $\lim_{x \rightarrow 2} x^3 - 3x + 9$

I $\lim_{x \rightarrow 5} [x]$

ANSWERS:

8 MY	4 DON	$\frac{1}{16}$ MONSTER	-1 THAT	$-\frac{1}{16}$ GOT
DNE I (not defined from right)	DNE YOU (not defined from left)	DNE 'VE (rt & left limits not equal)	8h AT	0 EMS
-8 BUT	16 ---	6 BL	$\frac{5}{3}$ A	5 THEY
2 WERS	$-\frac{5}{3}$ ME	-2 OWN	75 TH	$-\infty$ 'T
1 ER	$\frac{2}{3}$ BUT	∞ PRO	$-\frac{2}{3}$ ANS	7 BO

Answers

Limit Problems

1. $\frac{13}{2}$ 2. $\frac{11}{3}$ 3. $-\frac{1}{25}$ 4. ∞ 5. $\frac{1}{2\sqrt{2}}$ 6. 4 7. 12

8. -32 9. $\frac{3}{2}$ 10. 0 11. a) 6 b) 4 c) DNE 12. a) 1 b) 2 c) DNE

Limits BINGO

21	8	16	2	6
4	9	23	17	19
15	22	20	18	3
24	1	7	10	11
5	---	12	14	13

Puzzle:

What did one math book say to the other?

DON	'T	BO	TH	ER
ME	---	I	'VE	GOT
MY	OWN	PRO	BL	EMS

Answer: Don't bother me – I've got my own problems.